# CYCLIC SIEVING PHENOMENON IN NON-CROSSING CONNECTED GRAPHS

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Abstract. We prove an instance of the cyclic sieving phenomenon in non-crossing connected graphs, as conjectured by S.-P. Eu.

## 1. Introduction

A *non-crossing graph* on a finite set $S$ is a graph with vertices indexed by $S$ arranged in a circle such that no edges cross. When we say a graph on $n$ vertices, we will mean $S = \{1, \ldots, n\}$. In [3], Flajolet and Noy showed that the number $c_{n,k}$ of non-crossing connected graphs (see Figure 1) on $n$ vertices with $k$ edges, $n-1 \leq k \leq 2n-3$, is

$$c_{n,k} = \frac{1}{n-1}\binom{3n-3}{n+k}\binom{k-1}{n-2}. \tag{1}$$

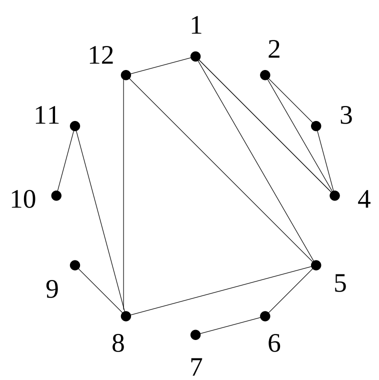


Figure 1. A non-crossing connected graph on 12 vertices with 14 edges

Define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

where $[n]!_q = [n]_q [n-1]_q \cdots [1]_q$ and $[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$. The formula in (1) admits a natural $q$-analogue:

$$c(n,k;q) = \frac{1}{[n-1]_q}\begin{bmatrix} 3n-3 \\ n+k \end{bmatrix}_q \begin{bmatrix} k-1 \\ n-2 \end{bmatrix}_q. \tag{2}$$

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It turns out that $c(n,k;q)$ is a polynomial in $q$, with nonnegative integer coefficients; see Proposition 6.1 below.

The main result of this paper is the following, which was conjectured by S.-P. Eu [1].

**Theorem 1.1.** *Let $n \geq 1$ and $n-1 \leq k \leq 2n-3$, and let $X$ be the set of non-crossing connected graphs on $n$ vertices with $k$ edges. If $d \geq 1$ divides $n$ and $\omega$ is a primitive $d$-th root of unity, then*

$$c(n,k;\omega) = s_d(n,k)$$

*where we define*

$$s_d(n,k) = \#\left\{x \in X : x \text{ is fixed under rotation by } \frac{2\pi}{d}\right\}.$$

In [4], Reiner, Stanton, and White introduced the notion of the cyclic sieving phenomenon. A triple $(X, X(q), C)$ consisting of a finite set $X$, a polynomial $X(q) \in \mathbb{N}[q]$ satisfying $X(1) = |X|$, and a cyclic group $C$ acting on $X$ exhibits the *cyclic sieving phenomenon* if, for every $c \in C$, if $\omega$ is a primitive root of unity of the same multiplicative order as $c$, then

$$X(\omega) = \#\{x \in X : c(x) = x\}.$$

In (1), the two extreme cases, $k = n-1$ and $k = 2n-3$, correspond to non-crossing spanning trees and $n$-gon triangulations respectively. In the former case, Eu and Fu showed in [2] that quadrangulations of a polygon exhibit the cyclic sieving phenomenon, where the cyclic action is cyclic rotation of the polygon, and they showed a bijection between quadrangulations of a $2n$-gon with non-crossing spanning trees on $n$ vertices. The bijection mapping is as follows: given a non-

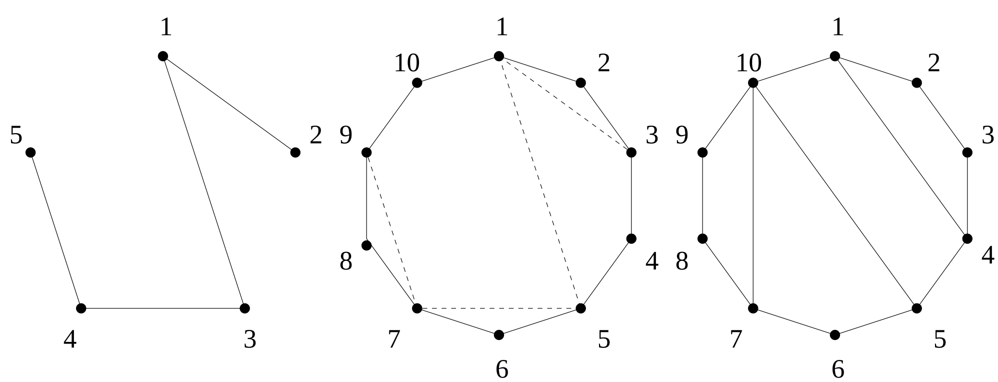


FIGURE 2. Bijection between a spanning tree on a 5 vertices and a quadrangulation of a 10-gon.

crossing spanning tree on $n$ vertices, for each edge connecting $i$ to $j$, draw a dotted line from $2i-1$ to $2j-1$ in a $2n$-gon. Then the quadrangulation of this $2n$-gon is defined by quadrangles whose diagonals are the dotted lines; conversely, given a $2n$-gon, every quadrangle has a diagonal whose endpoints are odd numbers, so we may perform the reverse procedure to get an inverse mapping (see Figure 2). This bijection preserves the cyclic sieving phenomenon, since rotation by $\frac{2\pi}{n}$ in the tree corresponds to rotation by $\frac{\pi}{n}$ in the $2n$-gon.

In the latter case, Reiner, Stanton, and White showed in [4] that polygon dissections of a polygon exhibit the cyclic sieving phenomenon where the cyclic action is also rotation. In particular, triangulations acted upon by rotations exhibit the cyclic sieving phenomenon. These results inspired Eu to conjecture Theorem 1.1, which we prove in the following sections. The case $d = 1$ in Theorem 1.1

follows from (1). We therefore consider the following three cases: $d = 2$ and $k$ is odd, $d = 2$ and $k$ is even, and $d \geq 3$.

## 2. Lagrange Inversion

In the following sections, we will use the Lagrange-Bürmann inversion theorem to extract coefficients of certain generating functions. If $\phi(z) \in \mathbb{Q}[[z]]$, then we define $[z^n]\phi(z)$ to be the coefficient of $z^n$ in $\phi(z)$.

**Lagrange inversion.** *Let $\phi(u) \in \mathbb{Q}[[u]]$ be a formal power series with $\phi(0) \neq 0$, and let $y(z) \in \mathbb{Q}[[z]]$ satisfy $y = z\phi(y)$. Then, for an arbitrary series $\psi$, the coefficient of $z^n$ in $\phi(y)$ is given by*

$$[z^n]\psi(y(z)) = \frac{1}{n}[u^{n-1}]\phi(u)^n\psi'(u).$$

Lagrange inversion may be applied to bivariate generating functions by treating the second variable as a parameter.

We begin by illustrating how Flajolet and Noy used Lagrange inversion to find (1). Let $C(z, w)$ be the generating function for $c_{n,k}$, that is,

$$C(z, w) = \sum_{n,k} c_{n,k} z^n w^k.$$

Then it can be shown using a combinatorial argument that $C$ satisfies

$$wC^3 + wC^2 - z(1 + 2w)C + z^2(1 + w) = 0.$$

Setting $C = z + zy$, this becomes

$$wz(1 + y)^3 = y(1 - wy)$$

which can be put in the Lagrange form

$$y = z\frac{w(1 + y)^3}{1 - wy}. \tag{3}$$

The result (1) then follows upon application of Lagrange inversion on $y$. We will in fact use this same function $y$ multiple times in our proofs.

## 3. The case where $d = 2$ and $k$ is odd

In this section, we prove that Theorem 1.1 holds when $d = 2$ and $k$ is odd. Recall that $d$ divides $n$, so $n$ must be even in this case. The case where $n = 2$ is trivial since there is only 1 non-crossing connected graph on 2 vertices, so we may assume that $n > 2$. For this section, define $n' = \frac{n}{2}$ and $k' = \frac{k+1}{2}$. It is a straightforward computation to verify that

$$c(n, k; -1) = \binom{3n' - 2}{n' + k' - 1}\binom{k' - 1}{n' - 1}. \tag{4}$$

Define

$$f_{n,k} = \#\{x \in X : x \text{ has an edge from 1 to } n\}.$$

**Lemma 3.1.** *With $f_{n,k}$ defined as above, we have*

$$s_2(n, k) = n' \cdot f_{n'+1,k'}.$$

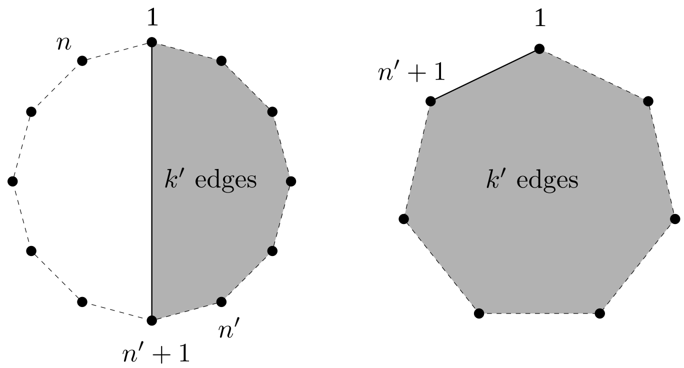


FIGURE 3. The bijection between centrally symmetric $n$-vertex, $k$-edge graph with fixed diameter and $(\frac{n}{2}+1)$-vertex, $\frac{k+1}{2}$-edge graph with edge between 1 and $\frac{n}{2}+1$.

*Proof.* Given a centrally symmetric with an odd number of edges, exactly one of the edges must be a diameter. There are $n'$ choices for the diameter. Once a diameter has been fixed, the remaining $k-1$ edges are determined by the $k'-1$ edges on either side of the diameter. Without loss of generality, assume the diameter has endpoints 1 and $n'+1$. Then we have a bijection (see Figure 3) between the graphs we wish to count and graphs on $\{1,\ldots,n'+1\}$ with $k'$ edges including the edge from 1 to $n'+1$. This is counted by $f_{n'+1,k'}$. □

We define some more notation. Recall that $c_{n,k} = |X|$. Define $d_{n,k}$ to be the number of non-crossing graphs on $\{1,\ldots,n\}$ with $k$ edges and exactly two connected components such that 1 and $n$ are in different components.

**Lemma 3.2.** *With $d_{n,k}$ defined above, we have*

$$d_{n,k} = \frac{2}{n-2}\binom{3n-5}{n+k}\binom{k-1}{n-3}.$$

*Proof.* Let $D(z,w) = \sum d_{n,k} z^n w^k$ and let $C(z,w) = \sum c_{n,k} z^n w^k$. Since $d_{n,k}$ counts graphs with two connected components, which are each counted by $c_{n,k}$, we therefore have $D = C^2$. To find the coefficient of $z^n w^k$, we use Lagrange inversion. Recall from (3) that $y = z\frac{w(1+y)^3}{1-wy}$. But $D = C^2 = z^2 + z^2(y^2+2y)$. Therefore

$$[z^n w^k]D = [z^{n-2}w^k]y^2 + 2[z^{n-2}w^k]y.$$

Computing each of these separately, we have

$$
\begin{aligned}
[z^{n-2}w^k]y &= \frac{1}{n-2}[u^{n-3}w^k]\frac{w^{n-2}(1+u)^{3n-6}}{(1-uw)^{n-2}} \\
&= \frac{1}{n-2}[u^{n-2}w^{k-n+2}]\frac{(1+u)^{3n-6}}{(1-uw)^{n-2}} \\
&= \frac{1}{n-2}(-1)^{k-n+2}\binom{2-n}{k-n+2}[u^{n-3}]u^{k-n+2}(1+u)^{3n-6} \\
&= \frac{1}{n-2}\binom{k-1}{k-n+2}[u^{n-3}]u^{k-n+2}(1+u)^{3n-6} \\
&= \frac{1}{n-2}\binom{k-1}{n-3}[u^{2n-k-5}](1+u)^{3n-6} \\
&= \frac{1}{n-2}\binom{3n-6}{n+k-1}\binom{k-1}{n-3}
\end{aligned}
$$

and

$$
\begin{aligned}
[z^{n-2}w^k]y^2 &= \frac{2}{n-2}[u^{n-4}w^k]\frac{w^{n-2}(1+u)^{3n-6}}{(1-uw)^{n-2}} \\
&= \frac{2}{n-2}[u^{n-4}w^{k-n+2}]\frac{(1+u)^{3n-6}}{(1-uw)^{n-2}} \\
&= \frac{2}{n-2}(-1)^{k-n+2}\binom{2-n}{k-n+2}[u^{n-4}]u^{k-n+2}(1+u)^{3n-6} \\
&= \frac{2}{n-2}\binom{k-1}{k-n+2}[u^{2n-k-6}](1+u)^{3n-6} \\
&= \frac{2}{n-2}\binom{3n-6}{n+k}\binom{k-1}{n-3}.
\end{aligned}
$$

where we have used the identity

$$\binom{n}{k} = (-1)^k\binom{k-n-1}{k}.$$

The result then follows from Pascal's identity. □

**Lemma 3.3.** *The sequence $f_{n,k}$ satisfies the recurrence*

$$f_{n,k} + f_{n,k+1} = c_{n,k} + d_{n,k}$$

*with the base case*

$$f_{n,2n-3} = c_{n,2n-3} = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

*Proof.* The base case follows from the fact that every triangulation must contain the edge from 1 to $n$. Now consider a non-crossing connected graph with $k+1$ edges on $\{1,\ldots,n\}$ with the edge 1 to $n$. We have two cases. When we remove this edge, either the remaining graph is connected or not. If the remaining graph is connected, then we have a non-crossing connected graph with $k$ edges without the edge from 1 to $n$. This is counted by $c_{n,k} - f_{n,k}$. If the remaining graph is not connected, then there are exactly two connected components, and 1 and $n$ lie in separate components. This is counted by $d_{n,k}$. Hence

$$f_{n,k+1} = c_{n,k} + d_{n,k} - f_{n,k}.$$

□

As a corollary to this lemma, it follows that one has the recurrence

$$s_2(2n-2,2k-1)+s_2(2n-2,2k+1)=(n-1)c_{n,k}+(n-1)d_{n,k} \tag{5}$$

with base case

$$s_2(2n-2,4n-7)=(n-1)c_{n,2n-3}=\binom{2n-4}{n-2}.$$

To show that $c(n,k;-1)=s_2(n,k)$ for even $n$ and odd $k$ or, equivalently, $c(2n-2,2k-1;-1)=s_2(2n-2,2k-1)$ for any positive integers $n>2$ and $n-1\le k\le 2n-3$, it suffices to show that $c(2n-2,2k-1;-1)$ satisfies the same recurrence (5) as $s_2(2n-2,2k-1)$. The base case is immediate:

$$c(2n-2,4n-7;-1)=\binom{3n-5}{3n-5}\binom{2n-4}{n-2}=\binom{2n-4}{n-2}.$$

We now show that $c(2n-2,2k-1;-1)$ satisfies the recurrence relation as well, which completes the proof that the theorem holds for $d=2$ and odd $k$.

**Proposition 3.4.** *$c(2n-2,2k-1;-1)$ satisfies*

$$c(2n-2,2k-1;-1)+c(2n-2,2k+1;-1)=(n-1)c_{n,k}+(n-1)d_{n,k}.$$

*Proof.* From (4), we see that all we need to verify is

$$\binom{3n-5}{n+k-2}\binom{k-1}{n-2}+\binom{3n-5}{n+k-1}\binom{k}{n-2}=\binom{3n-3}{n+k}\binom{k-1}{n-2}+\frac{2n-2}{n-2}\binom{3n-5}{n+k}\binom{k-1}{n-3},$$

which we leave as a straightforward exercise for the reader. □

## 4. The case where $d=2$ and $k$ is even

In this section, we prove that Theorem 1.1 holds when $d=2$ and $k$ is even. As in the previous case, it is again a straightforward computation to verify that

$$c(n,k;-1)=\binom{\frac{3n-4}{2}}{\frac{n+k}{2}}\binom{\frac{k-2}{2}}{\frac{n-2}{2}}.$$

Let $a_{n,k}$ denote the number of non-crossing connected graphs with $2n$ vertices and $k$ pairs of antipodal edges, where a diameter counts as one pair. Then

$$a_{n,k}=s_2(2n,2k-1)+s_2(2n,2k)=c(2n,2k-1;-1)+s_2(2n,2k).$$

Therefore, to show Theorem 1.1 holds for $d=2$ and even $k$, it suffices to show that

$$a_{n,k}=c(2n,2k-1;-1)+c(2n,2k;-1)=\binom{3n-1}{n+k}\binom{k-1}{n-1}. \tag{6}$$

Let $F$ be the generating function for $f_{n,k}$, i.e. $F(z,w)=\sum f_{n,k}z^nw^k$. Similarly, let $A$ be the generating function for $a_{n,k}$.

**Lemma 4.1.**

$$a_{n,k}=\sum_{m=1}^{n}\sum_{k_1+\cdots+k_m=k}\sum_{1\le v_1<\cdots<v_m\le n}\prod_{i=1}^{m}f_{v_{i+1}-v_i+1,k_i}$$

*where $v_{m+1}=v_1+n$.*

*Proof.* Consider a non-crossing connected graph with $2n$ vertices and $k$ pairs of antipodal edges. There exists a unique positive integer $m$ such that the center of the $2n$-gon lies inside a $2m$-gon formed by edges of the graph and such that no other edges lie inside the $2m$-gon. This $m$ is at most $n$. Now, exactly $m$ of the vertices of this $2m$-gon, call them $v_1 < \cdots < v_m$, lie in the set $\{1, \ldots, n\}$ due to the antipodal condition on the edges. All edges not used in the $2m$-gon lie outside

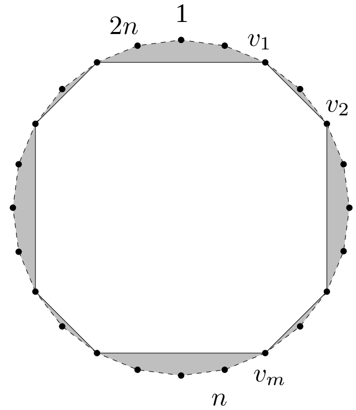


FIGURE 4. A graph with an inner $2m$-gon, where $m = 4$.

of it (see Figure 4). The $(m+1)$-th vertex is antipodal to $v_1$, hence $v_{m+1} = v_1 + n$. For each $i$, there is an edge from $v_i$ to $v_{i+1}$ and $k_i - 1$ other edges on the vertices $\{v_i, v_i + 1, \ldots, v_{i+1}\}$, such that $k_1 + \cdots + k_m = k$. Such a graph is counted by $f_{v_{i+1}-v_i+1,k_i}$. Thus we get the corresponding sum. □

**Lemma 4.2.** *With $A$ and $F$ as defined above, we have*

$$\frac{A}{z} = \frac{\partial(F/z)/\partial z}{1 - F/z}.$$

*Proof.* We show that

$$a_{n,k} = \sum_{m=1}^{n} \sum_{k_1+\cdots k_m=k} \sum_{n_1+\cdots+n_m=n+m} (n_m - 1) f_{n_m,k_m} \prod_{i=1}^{m-1} f_{n_i,k_i}.$$

In the sum in the previous lemma, the term $\prod_{i=1}^{m} f_{v_{i+1}-v_i+1,k_i}$ is counted multiple times with the product written in this order. We show that it is counted exactly $n + v_1 - v_m$ times. Consider any $m$-element subset $\{v_1, \ldots, v_m\} \subseteq \{1, \ldots, n\}$ with $v_1 < \cdots < v_m$. For $j = 1, \ldots, v_1 - 1$, this subset yields the same summand as $\{v_1 - j, \ldots, v_m - j\}$. Therefore, we can identify any subset $\{v_1, \ldots, v_m\}$ with $\{1, \ldots, v_m - v_1 + 1\}$. There are exactly $n + v_1 - v_m$ subsets corresponding to this one, each with largest element $v_m - v_1 + 1, v_m - v_1 + 2, \ldots, n$. This proves the sum identity above.

For the equality of generating functions, we insert variables into the above identity:

$$a_{n,k} z^n w^k = \frac{1}{z^{m-2}} \sum_{m=1}^{n} \sum_{k_1+\cdots k_m=k} \sum_{n_1+\cdots+n_m=n+m} (n_m - 1) f_{n_m,k_m} z^{n_m-2} w^{k_m} \prod_{i=1}^{m-1} f_{n_i,k_i} z^{n_i} w^{k_i}. \tag{7}$$

We note that

$$\frac{\partial(F/z)}{\partial z} = \sum_{n,k} (n-1) f_{n,k} z^{n-2} w^k$$

so, summing over all $n$ and $k$ in (7), we get

$$A = \frac{\partial(F/z)}{\partial z}\left(z + F + \frac{F^2}{z} + \frac{F^3}{z^2} + \cdots\right) = z\frac{\partial(F/z)}{\partial z}\left(\frac{1}{1-F/z}\right).$$

□

**Proposition 4.3.**

$$a_{n,k} = \binom{3n-1}{n+k}\binom{k-1}{n-1}.$$

*Proof.* Let $H = F/z$ and let $C$ be the generating function for $c_{n,k}$ as in the previous section and let $C = z + zy$. From the recurrence

$$f_{n,k} + f_{n,k+1} = d_{n,k} + c_{n,k}, \quad n \geq 2,$$

and

$$f_{1,k} = 0$$

we have

$$\left(1 + \frac{1}{w}\right)F = D + C - z = z^2(1+y)^2 + zy.$$

Therefore, after some substitution and simplification, applying the identity in (3), we get

$$1 - H = \frac{1}{1+y}.$$

From

$$\frac{A}{z} = \frac{\partial H/\partial z}{1-H}$$

we get

$$\int \frac{A}{z}dz = \int \frac{dH}{1-H}$$

or equivalently

$$\sum_{n,k}\frac{1}{n}a_{n,k}z^n w^k = -\log(1-H) = \log(1+y).$$

By Lagrange inversion,

$$\begin{aligned}
\frac{1}{n}a_{n,k} &= [z^n w^k]\int \frac{A}{z}\,dz \\
&= [z^n w^k]\log(1+y) \\
&= \frac{1}{n}[u^{n-1}w^k]\frac{w^n(1+u)^{3n}}{(1-uw)^n}\frac{1}{1+u} \\
&= \frac{1}{n}[u^{n-1}w^{k-n}]\frac{(1+u)^{3n-1}}{(1-uw)^n} \\
&= \frac{1}{n}(-1)^{k-n}\binom{-n}{k-n}[u^{n-1}]u^{k-n}(1+u)^{3n-1} \\
&= \frac{1}{n}\binom{k-1}{n-1}[u^{2n-k-1}](1+u)^{3n-1} \\
&= \frac{1}{n}\binom{3n-1}{n+k}\binom{k-1}{n-1}
\end{aligned}$$

whence our desired result. □

Comparing with (6) shows that Theorem 1.1 holds when $d = 2$ and $k$ is even.

## 5. The case where $d \geq 3$

Finally, in this section, we prove that Theorem 1.1 holds when $d \geq 3$. For this section, define $n' = \frac{n}{d}$ and $k' = \frac{k}{d}$. Again, it is a straightforward computation to verify that if $d|k$, then

$$c(n, k; \omega) = \binom{3n' - 1}{n' + k'}\binom{k' - 1}{n' - 1}.$$

**Lemma 5.1.** *If $d \geq 3$ does not divide $k$, then $c(n, k; \omega) = 0$, where $\omega$ is a primitive $d$-th root of unity.*

*Proof.* Suppose $k \equiv r \pmod{d}$, where $0 < r < d$. If $r > d - 3$, then $\begin{bmatrix} 3n-3 \\ n+k \end{bmatrix}_q = 0$. Grouping terms, we have

$$\begin{bmatrix} 3n - 3 \\ n + k \end{bmatrix}_q = \frac{\overbrace{[3n-3]_q \cdots}^{d-3}}{\underbrace{[n+k]_q \cdots}_{r}} \frac{\overbrace{[3n-d]_q \cdots [2n-k-d+r+1]_q}^{n+k-r}}{\underbrace{[n+k-r]_q \cdots [1]_q}_{n+k-r}} \overbrace{[2n-k-d+r]_q \cdots [2n-k-2]_q}^{r-d+3}.$$

The center block of $n + k - r$ ratios as well as the $d - 3$ ratios to the left of it go to some number in the limit as $q \to \omega$, and none of the terms to the left vanish at $q = \omega$ since none are divisible by $d$. However, $2n - k - d + r \equiv 0 \pmod{d}$ so $[2n - k - d + r]_{q=\omega} = 0$. Since $d - r \leq 2$, one has $2n - k - d + r \geq 2n - k - 2$ so the $[2n - k - d + r]_q$ term actually exists in the expression above. Similarly, if $r \leq d - 3$, then $\begin{bmatrix} k-1 \\ n-2 \end{bmatrix}_q = 0$. Grouping terms again, we have

$$\begin{bmatrix} k - 1 \\ n - 2 \end{bmatrix}_q = \frac{\overbrace{[k-1]_q \cdots}^{r-1}}{\underbrace{[n-2]_q \cdots}_{d-2}} \frac{\overbrace{[k-r]_q \cdots [k-r-n+d+1]_q}^{n-d}}{\underbrace{[n-d]_q \cdots [1]_q}_{n-d}} \overbrace{[k-r-n+d]_q \cdots [k-n+2]_q}^{d-r-1}.$$

As in the previous case, the center block of $n - d$ ratios as well as the $r - 1$ ratios to the left of it go to some number in the limit as $q \to \omega$, and none of the terms to the left vanish at $q = \omega$ since none are divisible by $d$. If $r < d - 3$, then $d - 2 \geq r$, so $k - r - n + d \geq k - n + 2$, hence the $[k - r - n + d]_q$ term exists in the expression above and since $k - r - n + d \equiv 0 \pmod{d}$, it has a zero at $q = \omega$. If $r = d - 3$, then there are exactly $d - r - 1 = 2$ terms in the right block, one of which is $[k - n + 3]_q$. Since $k - n + 3 \equiv r + 3 \equiv 0 \pmod{d}$, this term has a zero at $q = \omega$. $\square$

If $d$ does not divide $k$, then in fact there are no graphs with $k$ edges that are fixed under rotation by $\frac{2\pi}{d}$, since each edge lies in a free orbit under the action of rotation. We henceforth assume that $d|k$.

**Lemma 5.2.**

$$s_d(n, k) = n' \cdot f_{n'+1,k'} + s_2(2n', 2k').$$

*Proof.* If $\Gamma$ is a non-crossing connected graph on $\{1, \ldots, n\}$ fixed under rotation by $\frac{2\pi}{d}$, then there are two cases: either the edges form a central $d$-gon or not. In the former case, every edge is purely determined by the edges on the first $n' + 1$ vertices. In fact, there is bijection between such graphs and non-crossing connected graphs on $n' + 1$ vertices with the edge from 1 to $n' + 1$. There are $f_{n'+1,k'}$ such graphs, and there are $n'$ possible $d$-gons. In the latter case, the edges are determined by edges on the first $2n'$ vertices. We construct a bijection between such graphs and centrally symmetric non-crossing connected graphs on $2n'$ vertices with $2k'$ edges as follows (see

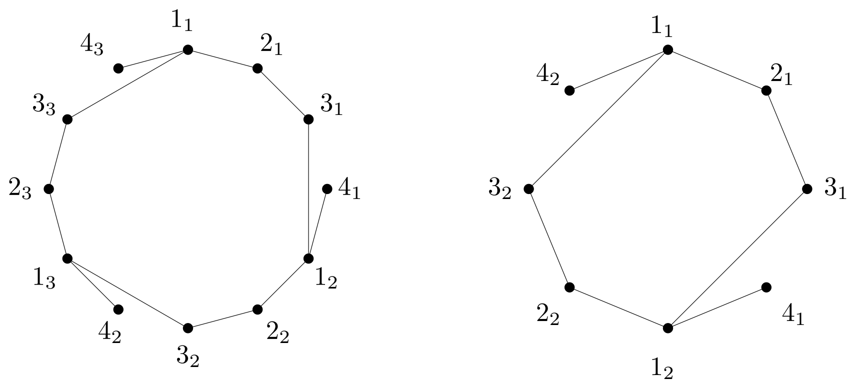


FIGURE 5. The bijective construction when $(n, k, d) = (12, 12, 3)$

Figure 5): Going around clockwise in the graph, label the first $m$ vertices $1_1, 2_1, \ldots, n'_1$, label the next set of vertices $1_2, 2_2, \ldots, n'_2$, and so on. Construct a non-crossing graph with $2n'$ vertices labeled $1_1, 2_1, \ldots, n'_1, 1_2, 2_2, \ldots, n'_2$. For each edge from $i$ to $j$ in the original graph, we put an edge with the same endpoints in the new graph. Finally, if there is an edge from some $i_2$ to some $j_3$, we put an edge from $i_2$ to $j_1$ in the new graph. This new graph therefore has $2k'$ edges. It is straightforward to check that this is a bijection. □

**Proposition 5.3.** *For $d \geq 3$ and $\omega$ a primitive $d$th root of unity,*

$$c(n, k; \omega) = s_d(n, k).$$

*Proof.* By the previous Lemma, and results from the case where $d = 2$,

$$\begin{aligned} s_d(n,k) &= n' \cdot f_{n'+1,k'} + s_2\left(2n', 2k'\right) \\ &= \binom{3n'-2}{n'+k'}\binom{k'-1}{n'-1} + \binom{3n'-2}{n'+k'-1}\binom{k'-1}{n'-1} \\ &= \left[\binom{3n'-2}{n'+k'} + \binom{3n'-2}{n'+k'-1}\right]\binom{k'-1}{n'-1} \\ &= \binom{3n'-1}{n'+k'}\binom{k'-1}{n'-1} \\ &= c(n,k;\omega). \end{aligned}$$

□

This completes the proof of Theorem 1.1.

## 6. Remarks and Future Work

Recall in the definition of the cyclic sieving phenomenon given in the Introduction that the function $X(q)$ must be a polynomial with nonnegative integer coefficients. Looking at (2), it is not *a priori* obvious that $c(n, k; q)$ is a polynomial with nonnegative integer coefficients. This is the content of our next Proposition.

**Proposition 6.1.** $c(n, k; q) \in \mathbb{N}[q]$.

*Proof.* To show that $c(n,k;q) \in \mathbb{Q}[q]$, it suffices to show that in the expansion

$$c(n,k;q) = \frac{\overbrace{[3n-3]_q \cdots [2n-k-2]_q}^{A}}{\underbrace{[n+k]_q \cdots [1]_q}_{B}} \frac{\overbrace{[k-1]_q \cdots [k-n+2]_q}^{C}}{\underbrace{[n-1]_q \cdots [2]_q}_{D}},$$

for each $j \geq 1$, if $\omega$ is a primitive $j$-th root of unity, then the order of the zero at $q = \omega$ is no smaller in the numerator than in the denominator. For $j = 1$, it is clear that the numerator and denominator each have a zero at $q = 1$ with multiplicity $2n + k - 2$. For $j = 2$, we have two cases. If $n$ is even, then block $B$ has $\lfloor \frac{n+k}{2} \rfloor$ zeros at $q = -1$ and block $D$ has $\frac{n-2}{2}$ zeros, whereas block $A$ has at least $\lfloor \frac{n+k}{2} \rfloor$ zeros at $q = -1$ and block $C$ has at least $\frac{n-2}{2}$ zeros. If $n$ is even, we have two subcases. If $k$ is odd, then the bottom has $\lfloor \frac{n+k}{2} \rfloor + \frac{n-1}{2}$ zeros while the the top has $\lceil \frac{n+k}{2} \rceil + \lfloor \frac{n-2}{2} \rfloor$, so they have the same order zero at $q = -1$. If $k$ is even, then both the top and bottom have a zero at $q = -1$ of order $\frac{n+k}{2} + \frac{n-1}{2}$. Now, if $j \geq 3$, we again have two cases. If $n - 1 \not\equiv 0 \pmod{j}$, then the bottom has a zero at $q = \omega$ of order $\lfloor \frac{n+k}{j} \rfloor + \lfloor \frac{n-1}{j} \rfloor$ while block $A$ has a zero of order $\geq \lfloor \frac{n+k}{j} \rfloor$ and block $C$ has a zero of order $\geq \lfloor \frac{n-1}{j} \rfloor$. If $n = 1 \equiv 0 \pmod{j}$, then the bottom has a zero of order $\lfloor \frac{n+k}{j} \rfloor + \frac{n-1}{j}$ while, since $3n - 3 \equiv 0 \pmod{j}$, the top has a zero of order $\geq \lceil \frac{n+k}{j} \rceil + \lfloor \frac{n-2}{j} \rfloor$ which is equal to $\lfloor \frac{n+k}{j} \rfloor + \lfloor \frac{n-2}{j} \rfloor + 1 = \lfloor \frac{n+k}{j} \rfloor + \frac{n-1}{j}$. It is well-known that $q$-binomial coefficients are polynomials in $\mathbb{N}[q]$ with symmetric unimodal coefficient sequences since, for example, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the generating function for integer partitions that fit in a $k \times (n-k)$ rectangle. Therefore,

$$\begin{bmatrix} 3n-3 \\ n+k \end{bmatrix}_q \begin{bmatrix} k-1 \\ n-2 \end{bmatrix}_q$$

is a product of polynomials in $\mathbb{N}[q]$ with symmetric unimodal coefficient sequences and is thus itself a polynomial in $\mathbb{N}[q]$ with a symmetric unimodal coefficient sequence. The fact that $c(n,k;q) \in \mathbb{N}[q]$ then follows from [4, Proposition 10.1]. $\square$

6.1. **Cyclic sieving phenomenon in other types of graphs.** In [3], one finds various formulas for counting classes of non-crossing graphs, of which (1) is one. Consider the following four formulae, found in [3]:

$$\begin{aligned}
T_n &= \frac{1}{2n-1}\binom{3n-3}{n-1} \\
C_{n,k} &= \frac{1}{n-1}\binom{3n-3}{n+k}\binom{k-1}{k-n+1} \\
D_{n,k} &= \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k} \\
P_{n,k} &= \frac{1}{n}\binom{n}{k}\binom{n}{k+1}
\end{aligned}$$

where $T_n$ is the number of non-crossing trees on $n$ vertices, $C_{n,k}$ is the number of non-crossing connected graphs on $n$ vertices, $D_{n,k}$ is the number of dissections of a convex $n$-gon using $k$ non-crossing diagonals, and $P_{n,k}$ is the number of non-crossing partitions of size $n$ with $n-k$ blocks (for a definition of a non-crossing partition, see [3, Section 3] or [4, Section 7.2]). These equations all

have natural $q$-analogues:

$$\begin{aligned} T(n;q) &= \frac{1}{[2n-1]_q}\begin{bmatrix} 3n-3 \\ n-1 \end{bmatrix}_q \\ C(n,k;q) &= \frac{1}{[n-1]_q}\begin{bmatrix} 3n-3 \\ n+k \end{bmatrix}_q \begin{bmatrix} k-1 \\ k-n+1 \end{bmatrix}_q \\ D(n,k;q) &= \frac{1}{[k+1]_q}\begin{bmatrix} n-3 \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \\ P(n,k;q) &= \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)}. \end{aligned}$$

Let $C$ be the cyclic group of order $n$ with a group action of rotation on a non-crossing graph on $n$ vertices. As mentioned in the Introduction, the collections of graphs for $T_n$, $D_{n,k}$, and $P_{n,k}$ exhibit the sieving phenomenon with respect to their respective $q$-analogues and $C$. Theorem 1.1 asserts that this is also true for those graphs counted by $C_{n,k}$ and $X(q) = C(n,k;q)$. However, there are more formulas in [3]:

$$\begin{aligned} F_{n,k} &= \frac{1}{2n-k}\binom{n}{k-1}\binom{3n-2k-1}{n-k} \\ G_{n,k} &= \frac{1}{n-1}\sum_{j=0}^{n-2}\binom{n-1}{k-j}\binom{n-1}{j+1}\binom{n-2+j}{n-2} \end{aligned}$$

where $F_{n,k}$ is the number of non-crossing forests on $n$ vertices with $k$ components and $G_{n,k}$ is the number of non-crossing (not necessarily connected) graphs on $n$ vertices with $k$ edges. These formulae admit $q$-analogues as well:

**Conjecture 6.2.** *Let $X$ be the set of non-crossing forests on $n$ vertices with $k$ components. Let*

$$X(q) = \frac{1}{[2n-k]_q}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} 3n-2k-1 \\ n-k \end{bmatrix}_q$$

*and let $C$ be the cyclic group of order $n$ acting on $X$ by rotation. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

**Conjecture 6.3.** *Let $X$ be the set of non-crossing graphs on $n$ vertices with $k$ edges. Let*

$$X(q) = \frac{1}{[n-1]_q}\sum_{j=0}^{n-2}\begin{bmatrix} n-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} n-1 \\ j+1 \end{bmatrix}_q \begin{bmatrix} n-2+j \\ n-2 \end{bmatrix}_q q^{j(j+n-k+2)}$$

*and let $C$ be the cyclic group of order $n$ acting on $X$ by rotation. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

6.2. **Unifying algebraic proof of cyclic sieving phenomenon in graphs.** So far, separate combinatorial proofs have been offered in [2], [4], and this paper for the exhibition of the cyclic sieving phenomenon in the context of the aforementioned classes of non-crossing graphs. It would be interesting to see an algebraic proof of Theorem 1.1 along the lines of [4, Proposition 2.1].

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